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Proceedings of the 5th International Conference on Inverse Problems in Engineering: Theory and Practice,Cambridge, UK, 11-15th July 2005

REMARKS ON COEFFICIENT DETERMINATION FOR THE STATIONARY ANISOTROPIC TRANSPORT EQUATION

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Abstract - The problem of simultaneous spatial determination of the absorption and scattering coefficients in the stationary linear one velocity Boltzmann transport equation via boundary measurements is investigated. The original first-order problem is shown to be equivalent to a second order self-adjoint problem. Then, I introduce an a priori operator K that can be different from the scattering but gives compactness to the problem. The associated eigenvalue problem generates a dense and complete set of eigenfunctions in the Hilbert space where the problem is defined. It is shown that the traces of eigenfunctions form a minimal system in the trace boundary space and that appropriate boundary values may be chosen in order to establish a bi-orthogonal set. Finally, the identifiability for the extinction and scattering coefficients is suggested in a very simplified way.

1. INTRODUCTION

We consider the boundary value problem for the linear stationary Boltzmann Transport equation

$$\begin{cases} \omega \cdot \nabla\phi(\omega,x) + q(x)\phi(\omega,x) - K_f[\phi](\omega,x) = 0 & \text{in } S\times\Omega\ ; \\ \phi(\omega,\sigma) = g^-(\omega,\sigma) & \text{on } \Sigma^-. \end{cases} \tag{1}$$

where Ω is a bounded and convex domain of $\mathbf{R}^N$, $N \geq 2$, $S = S^{N-1}$ denotes the unit sphere of $\mathbf{R}^N$, $\Sigma^{\pm} = \{(\omega,\sigma) \in S \times \partial\Omega : \pm\omega\cdot\nu(\sigma) > 0\}$ is the influx (outflux) boundary of $S\times\Omega$, $q(x)$ is the extinction coefficient due to absorption or scattering and K_f is the integral operator

$$K_f[\phi](t,\omega,x) = \int_S f(x,\omega'\cdot\omega)\phi(t,\omega',x)d\omega',$$

which describes the gain of particles in direction ω due to scattering from other directions. The function f is frequently expanded in an absolutely and uniformly convergent series

$$f(x,\omega\cdot\omega') = \sum_{k=1}^{\infty} \frac{2k-1}{4\pi} q_k(x) P_{k-1}(\omega\cdot\omega') \tag{2}$$

where P_{k-1} is the Legendre polynomials of degree $k-1$.

The combination of extinction and scattering defines the operator R^{-1}, $R^{-1}[\phi] = q(x)\phi - K_f[\phi]$ in $S\times\Omega$ which will be inverted for appropriate values of the coefficient q and f.

This solution ϕ defines the flux at the outflux boundary Σ^+

$$\phi = g^+ \text{on } \Sigma^+\ ;$$

and the Cauchy data for the problem

$$C_R = \{(g^-, g^+) \text{ on } \Sigma^- \times \Sigma^+\}$$

characterizes the graph for the albedo operator (the influx to outflux mapping) $\mathcal{A}_R$.

In the inverse problem we ask if it is possible to determine the coefficients of operator R, and the functions q and f from the a priori knowledge of the albedo operator. The problem is the investigation of the following mapping

$$\Phi : R \longrightarrow \mathcal{A}_R.$$

Note that when particle gain from scattering is neglected, that is $f = 0$, this is a transmission tomography problem, in which the Cauchy data (g^-, g^+) is the mathematical notation for the collimated source and detector data used in the x-ray reconstruction of the coefficient q [1]. In this generalized problem, additional coefficients q_k, $k = 1, 2, ...$ are to be reconstructed by measurements of non-collimated data which are usually neglected in the transmission problem. In reality, there are at least two-orders of magnitude between the two kind of data [6] and new technological strategies for the treatment of this problem are waiting for solutions.

In this work we will see that the first-order eqn.(1) is equivalent to a second order equation and this introduces a new series of questions similar to that found in the spectral analysis of the inverse scattering problem [4] and of the inverse heat transfer problem [14]. These analyses are all based on a second order self-adjoint partial differential equation when the problem is stationary, as in the case in study here. We will present here a spectral characterization of a second-order equivalent problem for eqn.(1), introducing a compact formulation and consequently avoiding a continuum spectrum. We also discuss a strategy for choosing boundary data (g^-, g^+) that generates a complete set of weight orthonornal functions dense in the Hibert space of solutions to problem (1) and how this can be used to theoretically ensure a unique reconstruction of the coefficients.

2. FUNCTIONAL FORMULATION

The appropriated setting for this kind of approach is the space of square integrable functions with support in the domain $S \times \Omega$, the space $H = L^2(S \times \Omega)$ in which the following scalar product is defined as

$$(\phi, \psi)_H = \int_{S\times\Omega} \phi(\omega, x)\psi(\omega, x)\ d\omega dx$$

Since the formulation introduced here is based on self-adjoint and compact operators, with real functions and discrete spectrum, we will restrict in this work to real functions spaces. In order to define the domain of the direction ω operator $A = \omega \cdot \nabla$, we will need the Hilbert subspace $W \subset H$, with internal scalar product

$$(\phi, \psi)_W = \int_{S\times\Omega} \{\phi(\omega, x)\psi(\omega, x) + A[\phi](\omega, x)A[\psi](\omega, x)\}d\omega dx$$

We note that

$$L^2(S; W^{1,2}(\Omega)) \subset W \subset H$$

with dense but not compact embedding. To define the boundary traces for solutions of problem (1) we will need the spaces

$$L^2(\Sigma) = L^2(\Sigma; |\omega \cdot \nu(\sigma)|d\omega d\sigma)$$

with scalar product

$$< \phi, \psi >_\Sigma = \int_\Sigma |\omega.\nu(\sigma)|\phi(\omega, \sigma)\psi(\omega, \sigma)d\omega d\sigma$$

and the space $\widetilde{W}$ with the following scalar product

$$< \phi, \psi >_{\widetilde{W}} = (\phi, \psi)_W + < \phi, \psi >_\Sigma .$$

The traces are well defined from $\widetilde{W}$ to $L^2(\Sigma)$, $L^2(\Sigma^+)$ or $L^2(\Sigma^-)$ [2] and so the operators $\gamma^\pm$ exist, are continuous and surjective, with $\gamma^\pm(\widetilde{W}) = L^2(\Sigma^\pm)$. In this situation the trace on the boundary surfaces $\Sigma^\pm$ has one right inverse which is the operator that translates values from the boundary to points inside the domain Ω

$$t_\pm[g](\omega, x) = g(\omega, x \pm \tau^\pm(\omega, x)\omega)$$

where $\tau^\pm(\omega, x) = sup\{t \in \mathbf{R}; x \pm t\omega \in \Omega\}$ is the distance from x to the boundary following the direction ω. The function $t_\pm[g]$ is continuous from $L^2(\Sigma^\pm)$ to $\widetilde{W}$ and satisfies the property

$$A[t_\pm[g]] = 0 \text{ for all } g \in L^2(\Sigma^\pm).$$

Another important fact is that the albedo operator for the problem (1) is adequately defined from $L^2(\Sigma^-)$ to $L^2(\Sigma^+)$.

To derive a second order theory which is equivalent to the first-order one, we need to define the unitary operator for inversion of direction in the Hilbert space H, $U[\phi](\omega, x) = \phi(-\omega, x)$ and the projection in H, $P = \frac{1}{2}(I + U)$ which decomposes H in two complementary subspaces

$$H^+ = \{\phi \in H : P(\phi) = \phi\} \text{ and } H^- = \{\phi \in H : P(\phi) = 0\}$$

and $\widetilde{W}$ in $\widetilde{W}^\pm = \widetilde{W} \cap H^\pm$.

We also need to define the inverse of R^{-1}, i.e. the operator

$$R = (qI + K_f)^{-1} : H \longrightarrow H \text{ with}$$

$$R[\phi](\omega, x) = \frac{1}{q(x)}\phi(\omega, x) + \int_S \sum_{k=1}^{\infty} \frac{2k-1}{4\pi} \frac{1}{q(x)} \frac{q_k(x)}{q(x) - q_k(x)} P_{k-1}(\omega' \cdot \omega)\phi(\omega', x)d\omega \tag{3}$$

where the coefficient functions q and q_k satisfy appropriate restrictions [10]. Sometimes it is also convenient to express the operator R by a divergent series kernel

$$R[\phi](\omega, x) = \int_S \sum_{k=1}^{\infty} \frac{2k-1}{4\pi} \frac{P_{k-1}(\omega' \cdot \omega)}{q(x) - q_k(x)} \phi(\omega', x)d\omega \tag{4}$$

3. FIRST AND SECOND ORDER PROBLEMS

We consider the boundary value problem (1) written in a formal operator form: find $\phi \in \widetilde{W}$ such that

$$T[\phi_1] = (A + R_1^{-1})[\phi 1] = h_1 \in H \tag{5}$$

$$\gamma_1^-[\phi] = g_1^- \in L^2(\Sigma^-) \tag{6}$$

which has a unique solution in $\widetilde{W}$ [5]. This solution defines the albedo operator

$$\mathcal{A}_R[g_1^-] = g_1^+ \in L^2(\Sigma^+). \tag{7}$$

$\mathcal{A}_R$ is linear and bounded with

$$|||\mathcal{A}_R||| = sup\{\| \mathcal{A}_R[g_1^-] \|_{L2(\Sigma+)}; \| g_1^- \| \leq 1\}$$

We also consider the adjoint boundary value problem: find $\phi_2 \in U[\widetilde{W}] = \widetilde{W}$ such that

$$T^*[\phi_2] = (-A + R_1^{-1})[\phi_2] = h_2 \in U[H] = H \tag{8}$$

$$\gamma_2^+[\phi] = g_2^+ \in L^2(\Sigma^+) \tag{9}$$

which has a unique solution in $\widetilde{W}$ [5]. This solution defines the adjoint albedo operator

$$\mathcal{A}_R^*[g_2^+] = g_2^- \in L^2(\Sigma^-) \tag{10}$$

with is also linear and bounded. For $R_1 = R_2$, that is, for direct and adjoint problems with the same coefficients, the operators $\mathcal{A}_R$ and $\mathcal{A}_R^*$ satisfy the following property

$$\int_{\Sigma^-} |\omega \cdot \nu(\sigma)| g_1^-(\omega, \sigma)\mathcal{A}_R^*[g_2^+](\omega, \sigma)d\omega d\sigma = \int_{\Sigma^+} |\omega \cdot \nu(\sigma)| g_2^+(\omega, \sigma)\mathcal{A}_R[g_1^-](\omega, \sigma)d\omega d\sigma \tag{11}$$

valid for every pair (g_1^-, g_2^+). If we choose an arbitrary $g_1^- = U[g_2^+] = g$ we obtain that

$$\mathcal{A}_R^* = U\mathcal{A}_R U : L^2(\Sigma^+) \longrightarrow L^2(\Sigma^-). \tag{12}$$

Now, for different coefficients, that is, $R_1 \neq R_2$, it is not difficulty to show that if ϕ_1 and ϕ_2 are solutions of (5) and (8), respectively, then

$$\begin{aligned} &\int_{S\times\Omega} (R_2^{-1} - R_1^{-2})[\phi_1](\omega, x)\phi_2(\omega, x)d\omega dx \\ &= \int_{\Sigma^+} |\omega \cdot \nu(\sigma)| [\mathcal{A}_{R_1}[g_1^-] - \mathcal{A}_{R_2}[g_1^-]](\omega, \sigma) g_2^+(\omega, \sigma)\ \omega d\sigma \\ &= \int_{\Sigma^-} |\omega \cdot \nu(\sigma)| [\mathcal{A}_{R_1}^*[g_2^+] - \mathcal{A}_{R_2}^*[g_2^+]](\omega, \sigma) g_1^-(\omega, \sigma)\ \omega d\sigma \end{aligned} \tag{13}$$

It is also not difficult to show that $T^* = UTU$, $-A = UAU$, $R = URU$ and $\gamma^+ = U\gamma^- U$ on theirs respective domains. We can formulate the homogeneous boundary values problem for problems (5) and (8), respectively, by adopting the following transformation

$$\psi_1(\omega, x) = \phi_1(\omega, x) - t_-[g_1^-](\omega, x) = \phi_1(\omega, x) - g_1^-(\omega, x - \tau^-(\omega, x)\omega) \tag{14}$$

$$\psi_2(\omega,x)=\phi_2(\omega,x)-t_+[g_2^+](\omega,x)=\phi_2(\omega,x)-g_2^-(\omega,x-\tau^-(\omega,x)\omega) \tag{15}$$

in eqns (5) and (8). In this case we obtain

$$T[\psi_1]=(A+R_1^{-1}[\psi_1])=h_1-R_1^{-1}[t_-[g_1^-]]\in H \tag{16}$$

$$\gamma^-[\psi_1]=0 \tag{17}$$

$$T^*[\psi_2]=(-A+R_2^{-1}[\psi_2])=h_2-R_2^{-1}[t_-[g_2^+]]\in U[H]=H \tag{18}$$

$$\gamma^+[\psi_2]=0 \tag{19}$$

where the fact that $A[t_\pm[g^\pm]]=0$ for all $g^\pm\in L(\Sigma^\pm)$ has been used.

To obtain the respective second order problems, we separate the non homogeneous internal source problem with homogeneous boundaries, (16) into a even and odd parity source problems ([5],[3],[10]) and after some manipulations we obtain an

$$(-AR_1A+R_1^{-1})[Pu_1]=P[h_1]\in H^+ \tag{20}$$

$$\gamma^-[(I-R_1A)[Pu_1]](\omega,\sigma)=\frac{1}{2}[g_1^-(\omega,\sigma)+g_1^-(-\omega,\sigma+\tau^+(\omega,\sigma)\omega)] \tag{21}$$

$$\gamma^+[(I+R_1A)[Pu_1]](\omega,\sigma)=\frac{1}{2}[g_1^-(-\omega,\sigma)+g_1^-(\omega,\sigma-\tau^-(\omega,\sigma)\omega)] \tag{22}$$

$$(-AR_1A+R_1^{-1})[(I-P)u_1]=(I-P)[h_1]\in H^- \tag{23}$$

$$\gamma^-[(I-R_1A)[(I-P)u_1]](\omega,\sigma)=\frac{1}{2}[g_1^-(\omega,\sigma)-g_1^-(-\omega,\sigma+\tau^+(\omega,\sigma)\omega)] \tag{24}$$

$$\gamma^+[(I+R_1A)[(I-P)u_1]](\omega,\sigma)=\frac{1}{2}[-g_1^-(-\omega,\sigma)+g_1^-(\omega,\sigma-\tau^-(\omega,\sigma)\omega)]. \tag{25}$$

Note that u_1 and R_1Au_1 are in $\widetilde{W}$ and $\phi_1=(I-R_1A)u_1$, that is $R_1^{-1}[\phi_1]=T_1^*[u_1]$.

We proceed in a similar way with the eqns (18) and obtain

$$(-AR_2A+R_2^{-1})[Pu_2]=P[h_2]\in H^+ \tag{26}$$

$$\gamma^+[(I+R_2A)[Pu_2]](\omega,\sigma)=\frac{1}{2}[g_2^+(\omega,\sigma)+g_2^+(-\omega,\sigma-\tau^-(\omega,\sigma)\omega)] \tag{27}$$

$$\gamma^-[(I-R_2A)[Pu_2]](\omega,\sigma)=\frac{1}{2}[g_2^+(-\omega,\sigma)+g_2^+(\omega,\sigma+\tau^+(\omega,\sigma)\omega)] \tag{28}$$

$$(-AR_2A+R_2^{-1})[(I-P)u_2]=(I-P)[h_2]\in H^- \tag{29}$$

$$\gamma^+[(I+R_2A)[(I-P)u_2](\omega,\sigma)=\frac{1}{2}[g_2^+(\omega,\sigma)-g_2^+(-\omega,\sigma-\tau^-(\omega,\sigma)\omega)] \tag{30}$$

$$\gamma^-[(I-R_2A)[(I-P)u_2]](\omega,\sigma)=\frac{1}{2}[-g_2^+(-\omega,\sigma)+g_2^+(\omega,\sigma+\tau^+(\omega,\sigma)\omega)]. \tag{31}$$

Also here u_2 and R_2Au_2 are in $\widetilde{W}$ and $\phi_2=(I+R_2A)u_2$, that is $R_2^{-1}[\phi_2]=T_2[u_2]$.

We have the following factorization for the second-order symmetric operator $L_R=-ARA+R^{-1}$,

$$L_R=R^{-1}(I+RA)(I-RA)=R^{-1}(I-RA)(I+RA)=T^*RT=TRT^*$$

Equations for the even (20) and odd parity (23) can be added in order to form a unique equation

$$L_{R_1}[u_1]=h_1=T_1[\phi_1]\in H \tag{32}$$

$$\gamma^-[(I+R_1A)u_1](\omega,\sigma)=g_1^-(\omega,\sigma)=\gamma^-[\phi_2] \tag{33}$$

$$\gamma^+[(I+R_1A)u_1](\omega,\sigma)=g_1^+(\omega,\sigma-\tau^-(\omega,\sigma)\omega))=\gamma^+[\phi_2](\omega,\sigma-\tau^-(\omega,\sigma)\omega)) \tag{34}$$

The adjoint equations with even (26) and odd parity (29) can receive a similar treatment to give

$$L_{R_2}[u_2]=h_2=T_2^*[\phi_2]\in H \tag{35}$$

$$\gamma^-[(I+R_2A)u_2](\omega,\sigma)=g_2^-(\omega,\sigma-\tau^-(\omega,\sigma)\omega))=\gamma^-[\phi_2](\omega,\sigma-\tau^-(\omega,\sigma)\omega) \tag{36}$$

$$\gamma^+[(I+R_2A)u_2](\omega,\sigma) = g_2^+(\omega,\sigma) = \gamma^-[\phi_2](\omega,\sigma) \tag{37}$$

The operator L_R defines a symmetric operator which is positive definite and, consequentially the Friedrichs theorem can be applied [12]. The

$$dom(L_R) = \{u \in \widetilde{W}; ARAu \in H; \gamma^-[(I-RA)u] = \gamma^+[(I+RA)u] = 0\}$$

can be completed in H with the energy scalar product

$$(u,v)_{H_R} = (RAu,v) + (R^{-1}u,v)_H + < u,v >_\Sigma$$

to form an energy Hilbert space in which the operator L_R is also self-adjoint.The associated norm can be proved to be equivalent to the norm defined in W for functions in the $dom(L_R)$ and with the norm in $\widetilde{W}$ for functions with nonhomogeneous traces. The formulation of the variational principle associated is straightforward and can be found in [3] and [5].

4. THE SPECTRAL PROBLEM

We can introduce different types of spectral problems associated with the stationary Boltzmann boundary value problem (1). The first one is the actual spectral problem derived from the transient form of the equation,which can lead to a non discrete complex spectrum [3]. The second procedure follows [5]. In order to explore the properties of the operator $L_q = A + qI$ and K_f, [5] shows that the combination $L_q^{-1}K_f$ forms a compact operator in H that can be left symmetrized to present a discrete real set of eigenvalues and eigenfunctions [8]. Whilst this is a good procedure in the solution of the direct problem, in the inverse parameter determination problem it presents a drawback from the fact that the operator K_f used in the symmetrization is not know a priori, since f is not known. We propose here an alternative approach based on the a priori knowledge of the operator $K : H \longrightarrow H$ which is supposed to be self-adjoint, positive and injective

$$K[\phi](\omega,.) = \int_S g(.,\omega'\cdot\omega)\phi(\omega',.)d\omega'$$

such that $K|_x : L^2(S) \longrightarrow L^2(S)$ is compact, a.e. $x \in \Omega$. We consider the following second order eigenvalue problem: find $u \in dom(L_R)$ such that

$$L_R u = \lambda K u \tag{38}$$

$$\int_{s\times\Omega} K[u](\omega,x)u(\omega,x)d\omega dx = 1 \tag{39}$$

It is not difficult to show that the operator $L_R^{-1}K$ is compact [11] and symmetrized by the operator K. It has an infinite set of real eigenvalues which are orthonormal for the following scalar product

$$(K[u_j],u_i)_H = \delta_{ji}. \tag{40}$$

From the injectivity of K, it is possible to show that there are an infinite set of eigenvalues $\{\lambda_j; j = 1,2,...\}$ and from symmetrization that there are no adjoint eigenfunctions, but each eigenvalue has multiple eigenfunctions. It can also be verified, with fundamental importance to the inverse problem, that the set of eigenfunctions is complete and dense in the range of the operator K, and consequently, in the space H. It is also not difficult to show that if u_j is an eigenfunction for the eigenvalue λ_j, then $U[u_j]$ is a eigenfunction for the same eigenvalue, that is, the two parity eigenfunctions

$$u_j^+ = Pu_j \text{ and } u_j^- = (I-P)u_j$$

satisfy the following eigenvalue problem

$$L_R u_j^\pm = \lambda_j K u_j^\pm \text{ on } S\times\Omega \tag{41}$$

$$\gamma^-[(I-RA)u_j^\pm] = \gamma^+[(I+RA)u_j^\pm] = 0$$

5. DENSE SETS OF EIGENFUNCTIONS

The eigenfunctions $\{u_j; j = 1, 2, ...\}$ for the eigenvalue problem

$$-ARAu_j + R^{-1}u_j = \lambda_j K u_j \text{ in } H \tag{42}$$

$$\gamma^-[(I - RA)u_j] = \gamma^+[(I + RA)u_j] = 0 \tag{43}$$

$$\int_{S\times\Omega} K[u_j](\omega, x)u_j(\omega, x)d\omega dx = 1 \tag{44}$$

establish in H an orthonormal set for the weight scalar product

$$(K[u_j], u_i)_H = \int_{S\times\Omega} K[u_j](\omega, x)u_i(\omega, x)d\omega dx = \delta_{ji} \text{ , } i, j = 1, 2, ... \tag{45}$$

Here the eigenvalues are repeated according to the multiplicity of their eigenfunctions. As we have noted, due to injectivity of the operator K, this set of functions is dense in H.

We are now in position to respond to the fundamental question for the identification problem, that is, if it is possible, by using only values of the flux at the influx boundary Σ^-, to generate a dense set such as the set of eigenfunctions $\{u_j; j = 1, 2, ...\}$. The answer for this question is affirmative and I will present the proof here.

Let us introduce the boundary spectral data for the pair (R, K), that is:

$$bsd(R, K) = \{(\lambda_j, \gamma^-[u_j]); j = 1, 2, ...\} \tag{46}$$

and the g-problem. Let g be an arbitrary function in $L^2(\Sigma^-)$ and $u_g \in \widetilde{W}$ the solution of

$$-ARAu_g + R^{-1}u_g = 0 \text{ in } H \tag{47}$$

$$\gamma^-[(I - RA)u_g](\omega, \sigma) = g(\omega, \sigma) \text{ if } (\omega, \sigma) \in \Sigma^- \tag{48}$$

$$\gamma^-[(I + RA)u_g](\omega, \sigma) = g(-\omega, \sigma + \tau_+(\omega, \sigma)\omega) \text{ if } (\omega, \sigma) \in \Sigma^+ \tag{49}$$

We can obtain an explicit generalized Fourier series for the solution u_g by multiplying eqn. (47) by u_j, integrate on $S \times \Omega$ and use (42),(43),(48) and (49) in a straightforward way to obtain

$$\lambda_j \int_{S\times\Omega} K[u_j](\omega, x)u_g d\omega dx =$$

$$= \int_{\Sigma^-} |\omega \cdot \nu(\omega)|g(\omega, \sigma)u_j(\omega, \sigma)d\omega d\sigma + \int_{\Sigma^+} |\omega.\nu(\sigma)|g(-\omega, \sigma + \tau^+(\omega, \sigma)\omega)d\omega d\sigma$$

We first note that $\{(K[u_g], u_j)_H; j = 1, 2, ...\}$ are the general Fourier coefficients for the function u_g with respect to the systems of functions $\{u_j; j = 1, 2, ...\}$ and that the series

$$K[u_g] = \sum_{j=1}^{\infty}(K[u_j], u_g)_H K[u_j] \in H$$

converges in the norm of H. In this way the formal solution to the g-problem based on this complete system of eigenfunctions is

$$u_g = \sum_{j=1}^{\infty}(\frac{1}{\lambda_j}\int_{\Sigma} |\omega.\nu(\sigma)|\Psi(\omega, \sigma)u_j(\omega, \sigma)d\omega d\sigma)u_j \tag{50}$$

where we have defined

$$\Psi(\omega, \sigma) = \begin{cases} g(\omega, \sigma) \text{ if } (\omega, \sigma) \in \Sigma^-; \\ g(-\omega, \sigma + \tau^+(\omega, \sigma)\omega) \text{ if } (\omega, \sigma) \in \Sigma^+. \end{cases} \tag{51}$$

Note that for a pair of operators (R, K), R with coefficients guessed from the identification problem and K choosed in an such way that gives completeness of the derived system of eigenfunctions (at least injective), we can choose a set $\{g_i; i = 1, 2, ...\}$ in a such way that the pairs $\{(\Psi_i, u_j); i, j = 1, 2, ...)\}$ form a bi-orthonormal system of functions, [13], in the Hibert space $L^2(\Sigma; |\omega \cdot \nu(\sigma)|d\omega d\sigma))$, that is,

$$< \Psi_i, u_j >_\Sigma = \int_\Sigma |\omega.\nu(\sigma)|\Psi_i(\omega,\sigma)u_j(\omega,\sigma) = \delta_{ij} \ , i, j = 1, 2, ... \tag{52}$$

Then the respective g_i-problem will have formal solution

$$u_{g_j} = \frac{1}{\lambda_j} u_j, j = 1, 2, ...$$

and the completeness of the system u_j is transferred to the system u_{g_j}. In fact, the system of functions in the boundary spectral data, that is, the traces on $L^2(\Sigma; |\omega \cdot \nu(\sigma)|d\omega d\sigma)$ of the functions $\{u_j; j = 1, 2, ...\}$ is minimal in the sense of Lewin, Kaczmarz and Steeinhaus [13] and this is the necessary and sufficient condition for the existence of the bi-orthonormal system $\{\Psi_i; i = 1, 2, ...\}$. It is not difficult to see this, since if an eigenvalue λ_j has multiplicty m_j, then the m_j's eigenfunctions $\{u_{jm}; m = 1, 2, ..., m_j\}$ associated with it have trace on $L^2(\Sigma; |\omega \cdot \nu(\omega)|d\omega d\sigma)$ and span a subspace with dimension exactly equal to m_j. Systems of functions linearly independent in the bulk space $\widetilde{W}$ are linearly independent in the trace boundary space $L^2(\Sigma; |\omega \cdot \nu(\sigma)|d\omega d\sigma)$ by the continuation property. We finally show by using parity properties that this set can be constructed with functions $\{\Psi_j; j = 1, 2, ...\}$ related with g_j's by formula (51). Note that

$$P[\Psi_j](\omega,\sigma) = \begin{cases} \frac{1}{2}[g_j(\omega,\sigma) + g_j(-\omega, \sigma + \tau^+(\omega,\sigma)\omega) & \text{if } (\omega,\sigma) \in \Sigma^-; \\ \frac{1}{2}[g_j(-\omega,\sigma) + g_j(\omega, \sigma - \tau^-(\omega,\sigma)\omega) & \text{if } (\omega,\sigma) \in \Sigma^+ \end{cases} \tag{53}$$

and

$$(I-P)[\Psi_j](\omega,\sigma) = \begin{cases} \frac{1}{2}[g_j(\omega,\sigma) - g_j(-\omega, \sigma + \tau^+(\omega,\sigma)\omega) & \text{if } (\omega,\sigma) \in \Sigma^-; \\ \frac{1}{2}[-g_j(-\omega,\sigma) + g_j(\omega, \sigma - \tau^-(\omega,\sigma)\omega) & \text{if } (\omega,\sigma) \in \Sigma^+ \end{cases} \tag{54}$$

and that the boundary space $L^2(\Sigma; |\omega \cdot \nu(\sigma)|d\omega d\sigma)$ is decomposed by parity into two complementary subspaces with the scalar product (52) rewritten as

$$< \Psi_i, u_j >_\Sigma = \int_\Sigma |\omega.\nu(\sigma)|(P[\Psi_i](\omega,\sigma)P[u_j](\omega,\sigma) + (I-P)[\Psi_i](\omega,\sigma)(I-P)[u_j](\omega,\sigma))d\omega d\sigma \tag{55}$$

since

$$\int_\Sigma |\omega.\nu(\sigma)|P[\Psi_i](\omega,\sigma)(I-P)[u_j](\omega,\sigma)d\omega d\sigma = \int_\Sigma |\omega.\nu(\sigma)|(I-P)[\Psi_i](\omega,\sigma)P[u_j](\omega,\sigma)d\omega d\sigma = 0.$$

We see that we can choose different g_j's to generate the even and odd parity functions separately, say, a g_{j_1} to obtain the even parity bi-orthonormality, and a g_{j_2} for the odd parity, and then add the result to generate the functions $\{\Psi_j; j = 1, 2, ...\}$.

6. IDENTIFIABILITY OF COEFFICIENTS

From the eqn.(13)and the fact the first order direct and adjoint problems are related to their respective second-order problems by $\phi_1 = (I - RA)u_1$ and $\phi_2 = (I + RA)u_2$ we obtain the basic identity to be used in the identification problem

$$\int_\Omega (q_2(x) - q_1(x)) \int_S [(I - R_1A)u_1][(I + R_2A)u_2]d\omega dx$$

$$- \int_{S\times\Omega} (\int_S [f_2(x, \omega' \cdot \omega) - f_1(x, \omega' \cdot \omega)][(I - R_1A)u_1](\omega', x)d\omega')[(I + R_2A)u_2]d\omega dx$$

$$= \int_{\Sigma^+} |\omega \cdot \nu(\sigma)|[\mathcal{A}_{R_1}[g_1] - \mathcal{A}_{R_2}[g_1]](\omega,\sigma)g_2(\omega,\sigma)d\omega d\sigma = 0 \tag{56}$$

The two problems (1 = direct and 2 = adjoint) by hypothesis have possibly different coefficients with the same albedo, and so the right hand-side of this equality is zero, independently of the data g_1 and g_2. By using the property $R_1^{-1} - R_2^{-1} = R_2^{-1}(R_2 - R_1)R_1^{-1}$, the left hand side of eqn.(56) can be rewritten as

$$\int_{S\times\Omega} (R_2 - R_1)[T_1[u_1]](\omega, x)T_2^*[u_2](\omega, x)d\omega dx = 0 \tag{57}$$

or, explicitly thought the divergent series (4)

$$\int_{S\times\Omega}\int_S(\sum_{l=1}^{\infty}\frac{2l-1}{4\pi}(\frac{1}{q_2(x)-q_{2l}(x)}-\frac{1}{q_1(x)-q_{2l}(x)})P_{l-1}(\omega'\cdot\omega))$$

$$T_1[u_1](\omega',x)T_2^*[u_2](\omega,x)d\omega' d\omega dx=0 \tag{58}$$

Since the solutions u_1 and u_2 are arbitrary and can be made to range over a complete set of linearly independent functions in $\widetilde{W}$, we see that the two operators must be equal with coefficients $\{\frac{1}{q(x)-q_l(x)}\in L^\infty(\Omega); l=1,2,...\}$. Since the sequence $\{q_l\}$ converges asymptotically to zero, the two extinction coefficients, q_1 and q_2, are also equal.

7. CONCLUSIONS

The simultaneous spatial determination of the absorption and scattering coefficients in the stationary linear one velocity Boltzmann transport equation via boundary measurements is showed to be possible in a second order self-adjoint compact spectral problem. An eigenvalue problem associated can be implemented to generate a dense and complete set of eigenfunctions in the Hilbert space where the problem is defined. The traces of eigenfunctions form a minimal system in the trace boundary space and appropriate boundary values may be chosen in order to establish a bi-orthogonal set used to show the identifiability of the coefficients.

Acknowledgement

The author would like to acknowledge some financial support received from CNPq/Brazil and Project CAPES-GRICES.

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